$GT_{3\frac{1}{2}}$-spaces, I

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Abstract

In 2001 the authors had introduced different L-separation axioms denoted by GT_i, $i = 0,1,2,3,4$. In this and a subsequent paper, we introduce a notion of completely regular L-topological spaces depend on the notion of L-numbers presented by S. Gähler and W. Gähler in 1994. We denote by $GT_{3\frac{1}{2}}$-space (or L-Tychonoff space) the L-topological space which is GT_1 and completely regular in this sense. The category L**-Tych** of $GT_{3\frac{1}{2}}$-spaces is topological over the category **Set** of sets, that is, the initial and final lifts and also the initial and final $GT_{3\frac{1}{2}}$-spaces exist in L**-Tych**. Moreover, the relation between the $GT_{3\frac{1}{2}}$-spaces, GT_4-spaces and GT_3-spaces, goes well. It is also shown here that our completely regular spaces are more general than the completely regular spaces defined by Hutton in 1975, Katsaras in 1980, and by Kandil and El-Shafee in 1988. The relation of the $GT_{3\frac{1}{2}}$-spaces with the L-proximity spaces, the L-uniform spaces and the L-compact spaces will be investigated in part II of this paper. Moreover, the relation between the $GT_{3\frac{1}{2}}$-spaces and the L-topological groups will be studied in a separate paper.

Keywords: Fuzzy filters; GT_i-spaces; completely regular spaces; $GT_{3\frac{1}{2}}$-spaces; L-Tychonoff spaces; Initial and final lifts; Initial and final L-topological spaces; L-proximity spaces; L-uniform spaces; L-compact spaces; L-topological groups.

Introduction

There is a notion of L- real numbers introduced by S. Gähler and W. Gähler in [10], and defined as a convex, normal, compactly supported and upper semi-continuous L-subsets of the set of real numbers $\mathbf{R}$. The set of all L-real numbers is called

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L-L-*real line* and is denoted by $\mathbf{R}_L$, where L is a complete chain.

In this paper, using the space $(I_L, \Im)$, where $I = [0,1]$ is the closed unit interval and $\Im$ is the L-topology on I_L, a notion of completely regular L-topological spaces is introduced. This completely regular L-topological space is defined, as in case of GT_i-spaces, $i = 0,1,2,3,4$, using the ordinary points and usual subsets. The L-topological space which is GT_1 and completely regular in our sense will be denoted here by $GT_{3\frac{1}{2}}$-space (or L-Tychonoff space) and the category of all $GT_{3\frac{1}{2}}$-spaces will be denote by L-**Tych**. For these $GT_{3\frac{1}{2}}$-spaces, the Urysohn Lemma is proved and hence it is shown that each GT_4-space is a $GT_{3\frac{1}{2}}$-space. Moreover, each $GT_{3\frac{1}{2}}$-space is a GT_3-space. For each case a counter example will be given. It is also shown that the $GT_{3\frac{1}{2}}$-space is an extension with respect to the functor ω, defined by Lowen in [23], from the category **Tych** of $T_{3\frac{1}{2}}$-spaces to the category L-**Tych**.

The category L-**Tych** is topological over the category **Set** of sets ([1]). This means that the initial and the final of a family of $GT_{3\frac{1}{2}}$-spaces also are $GT_{3\frac{1}{2}}$-spaces. As special initial and final L- topological spaces, the subspace, the product space, the quotient space and the sum space of $GT_{3\frac{1}{2}}$-spaces are $GT_{3\frac{1}{2}}$-spaces.

There are several notions of completely regular L-topological spaces such as the notions defined by Hutton in [16], by Katsaras in [20] and by Kandil and El-Shafee in [17]. In the last section, it is shown that our notion of completely regular L-topological spaces is more general than these notions ([16, 17, 20]). Counter examples are given to show these generalizations.

1. Preliminaries

Throughout the paper let L be a complete chain with different least and last elements 0 and 1, respectively. Let $L_0 = L \setminus \{0\}$, $L_1 = L \setminus \{1\}$ and $I_{01} = I \setminus \{0,1\}$, where I is the closed unit interval. L^X and $P(X)$ denote the sets of all L-subsets, that is all mappings $f : X \to L$, and of all ordinary subsets of X, respectively. Assume that an order-reversing involution $\alpha \mapsto \alpha'$ of L is fixed. For each L-set $f \in L^X$, let f'

denote the complement of f, defined by $f'(x) = f(x)'$ for all $x \in X$. $\sup f$ denotes the supremum of values of f. For all $f, g \in L^X$, f is called *quasi-coincident* with g, denoted by $f\, q\, g$, if $f \not\leq g'$. Denote by $\overline{q}$ the not quasi-coincident, that is, $f\, \overline{q}\, g$ means $f \leq g'$. For each $x \in X$ and $\alpha \in L_0$, let x_α denote an L-point in X. For each $\alpha \in L$, the constant L-subset of X with value α will be denoted by $\overline{\alpha}$. For each L-set $f \in L^X$, and $\alpha \in L$ let $s_\alpha f = \{x \in X \mid f(x) > \alpha\}$ and $w_\alpha f = \{x \in X \mid f(x) \geq \alpha\}$ be the strong α-cut and the weak α-cut of f, respectively ([22]).

L-topological spaces. In the following the L-topology τ on a set X in sense of [7, 15] will be used. τ is called *stratified* if $\overline{\alpha} \in \tau$ for all $\alpha \in L$. Let τ' denote the set of all closed L-sets in (X, τ). Denote by $\text{int}_\tau f$ and $\text{cl}_\tau f$ the interior and the closure of an L-subset f of X with respect to τ, respectively. Let (X, τ) and (Y, σ) be two L-topological spaces. Then the mapping $f : (X, \tau) \to (Y, \sigma)$ is called *L-continuous* (or *(τ, σ)-continuous*) provided $\text{int}_\sigma g \circ f \leq \text{int}_\tau (g \circ f)$ for all $g \in L^Y$. If T is an ordinary topology on X, then the *induced L-topology* ([23]) on X is given by.

$$\omega(T) \;=\; \{f \in L^X \mid s_\alpha f \in T \text{ for all } \alpha \in L_1\}.$$

Initial and final L-topological spaces. Consider a family of L-topological spaces $((X_i, \tau_i))_{i \in I}$. Let $\bigvee\limits_{i \in I} f_i^{-1}(\tau_i)$ be the supremum of the family $(f_i^{-1}(\tau_i))_{i \in I}$, where for each $i \in I$, $f_i^{-1}(\tau_i) = \{f_i^{-1}(g) \mid g \in \tau_i\}$ and f_i is a mapping of a set X into sets X_i. Moreover, let $\bigwedge\limits_{i \in I} f_i(\tau_i)$ be the infimum $\bigwedge\limits_{i \in I} f_i(\tau_i)$ of the family $(f_i(\tau_i))_{i \in I}$, where $f_i(\tau_i) = \{g \in L^X \mid f_i^{-1}(g) \in \tau_i\}$ and f_i is a mapping of X_i into X. Of course these supremun and infimum are taken with respect to the finer relation on L-topologies. $\bigvee\limits_{i \in I} f_i^{-1}(\tau_i)$ and $\bigwedge\limits_{i \in I} f_i(\tau_i)$ fulfill the following result.

Proposition 1.1 [5, 22] *$\bigvee\limits_{i \in I} f_i^{-1}(\tau_i)$ and $\bigwedge\limits_{i \in I} f_i(\tau_i)$ are the initial and the final of $(\tau_i)_{i \in I}$ with respect to $(f_i)_{i \in I}$, respectively.*

It is known that, the initial (final) L-topology of $(\tau_i)_{i \in I}$ with respect to $(f_i)_{i \in I}$ is the L-topology τ on X which provides an initial (a final) lift in the category ***L*-Top** of L-topological spaces. That is, all mappings $f_i : (X, \tau) \to (X_i, \tau_i)$ $(f_i : (X_i, \tau_i) \to$

$(X,\tau))$ are L-continuous and for any L-topological space (Y,σ) and mapping $f : Y \to X$ ($f : X \to Y$), $f : (Y,\sigma) \to (X,\tau)$ ($f : (X,\tau) \to (Y,\sigma)$) is L-continuous if and only if for all $i \in I$ the mappings $f_i \circ f : (Y,\sigma) \to (X_i,\tau_i)$ ($f \circ f_i : (X_i,\tau_i) \to (Y,\sigma)$) are L-continuous ([1, 5]).

L-filters. By an *L-filter* on X ([9, 11]) is meant a mapping $\mathcal{M} : L^X \to L$ such that the following conditions are fulfilled.

(F1) $\mathcal{M}(\overline{\alpha}) \leq \alpha$ holds for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$.

(F2) $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$.

An L-filter $\mathcal{M}$ is called *homogeneous* if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. For each $x \in X$, the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(f) = f(x)$ for all $f \in L^X$ is a homogeneous L-filter on X. For L-filters $\mathcal{M}$ and $\mathcal{N}$ on X, $\mathcal{M}$ is said to be *finer than* $\mathcal{N}$, denoted by, $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(f) \geq \mathcal{N}(f)$ holds for all $f \in L^X$. By $\mathcal{M} \nleq \mathcal{N}$ we denote that $\mathcal{M}$ is not finer than $\mathcal{N}$. For all L-filters $\mathcal{M}, \mathcal{N}, \mathcal{L}$ on X, we have ([2])

$$\mathcal{L} \geq \mathcal{M} \nleq \mathcal{N} \text{ implies } \mathcal{L} \nleq \mathcal{N}. \tag{1.1}$$

For each non-empty set A of L-filters on X, the supremum $\bigvee\limits_{\mathcal{M} \in A} \mathcal{M}$ with respect to the finer relation of L-filters exists and we have

$$\Big(\bigvee_{\mathcal{M} \in A} \mathcal{M}\Big)(f) = \bigwedge_{\mathcal{M} \in A} \mathcal{M}(f)$$

for all $f \in L^X$ ([9]). The infimum $\bigwedge\limits_{\mathcal{M} \in A} \mathcal{M}$ doesn't exist in general, and the following proposition gives the condition of the existence of this infimum.

Proposition 1.2 [9] *For any set A of L-filters on X the infimum $\bigwedge\limits_{\mathcal{M} \in A} \mathcal{M}$ of A exists* if and only if *for each non-empty finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of A we have $\mathcal{M}_1(f_1) \wedge \cdots \wedge \mathcal{M}_n(f_n) \leq \sup(f_1 \wedge \cdots \wedge f_n)$ for all $f_1, \ldots, f_n \in L^X$. If the infimum of A exists, then for each $f \in L^X$ and n as a positive integer we have:*

$$\Big(\bigwedge_{\mathcal{M} \in A} \mathcal{M}\Big)(f) = \bigvee_{\substack{f_1 \wedge \cdots \wedge f_n \leq f, \\ \mathcal{M}_1, \ldots, \mathcal{M}_n \in A}} (\mathcal{M}_1(f_1) \wedge \cdots \wedge \mathcal{M}_n(f_n)).$$

L-neighborhood filters. For each L-topological space (X, τ) and each $x \in X$ the L-filter $\mathcal{N}(x)$ on X, defined by $\mathcal{N}(x)(f) = \operatorname{int}_\tau f(x)$ for all $f \in L^X$, is called the *L-neighborhood filter* of the space (X, τ) at x, We may note that $\dot{x} \leq \mathcal{N}(x)$ holds for all $x \in X$ ([12]). The L-neighborhood filter $\mathcal{N}(F)$ at an ordinary subset F of X is the L-filter on X defined, by the authors in [3], by means of $\mathcal{N}(x)$, $x \in F$ as: $\mathcal{N}(F) = \bigvee_{x \in F} \mathcal{N}(x)$. Moreover, the L-filter $\dot{F}$ is defined by $\dot{F} = \bigvee_{x \in F} \dot{x}$ and we have $\dot{F} \leq \mathcal{N}(F)$ holds for all $F \in P(X)$. Recall also here the L-filter $\dot{g}$ and the L-neighborhood filter $\mathcal{N}(g)$ at an L-subset g of X defined by ([6])

$$\dot{g}(f) = (\bigvee_{0 < g(x)} \dot{x})\,(f) \quad \text{and} \quad \mathcal{N}(g)(f) = (\bigvee_{0 < g(x)} \mathcal{N}(x))\,(f), \tag{1.2}$$

respectively, for all $f \in L^X$. We have $\dot{g} \leq \mathcal{N}(g)$ holds for all $g \in L^X$.

GT_i-spaces. In [2, 3] we had defined the L-separation axioms GT_i, $i = 0, 1, 2, 3, 4$, and in the following we recall some of these axioms which are needed in this paper. An L-topological space (X, τ) is called:

(1) GT_0 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\leq \mathcal{N}(y)$ or $\dot{y} \not\leq \mathcal{N}(x)$.

(2) GT_1 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\leq \mathcal{N}(y)$ and $\dot{y} \not\leq \mathcal{N}(x)$.

(3) GT_2 if for all $x, y \in X$ with $x \neq y$ we have $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist.

(4) *regular* if $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist for all $x \in X, F \in P(X)$ with $F \in \tau'$ and $x \notin F$.

(5) GT_3 if it is regular and GT_1.

(6) *normal* if for all $F_1, F_2 \in P(X)$ with $F_1, F_2 \in \tau'$ and $F_1 \cap F_2 = \emptyset$ we have $\mathcal{N}(F_1) \wedge \mathcal{N}(F_2)$ does not exist.

(7) GT_4 if it is normal and GT_1.

Denote by *GT_i-space* the L-topological space which is GT_i.

Proposition 1.3 [2, 3] *We have the following results:*

(1) *Every GT_i-space is GT_{i-1}-space for each $i = 1, 2, 3, 4$.*

(2) *The initial and final L-topological spaces of a family of GT_i-spaces are GT_i-spaces for each $i = 0, 1, 2, 3, 4$.*

(3) *The L-topological subspace and the L-topological product space of a family of GT_i-spaces are GT_i-spaces for each $i = 0, 1, 2, 3, 4$.*

In [4] we had studied the relation of GT_i-spaces with other notions of L-separation axioms, and in [6] we had studied also the relation of GT_i-spaces with the L-proximity spaces defined in [19], the G-compact spaces defined in [12] and the L-uniform spaces defined in [14].

L-real numbers. The $GT_{3\frac{1}{2}}$-spaces will be defined in Section 2 using the Gähler's L-unit interval I_L. By *Gähler's L-unit interval* ([10]) is meant the set I_L defined by

$$I_L \;=\; \{x \in \mathbf{R}_L^* \mid x \leq 1^\sim\},$$

where $I = [0, 1]$ is the real unit interval and $\mathbf{R}_L^* = \mathbf{R}_L^* = \{x \in \mathbf{R}_L \mid x(0) = 1 \text{ and } 0^\sim \leq x\}$ is the set of all positive L-real numbers. Note that here, by $\leq$ we mean the binary operation on $\mathbf{R}_L$ defined by

$$x \leq y \Leftrightarrow x_{\alpha_1} \leq y_{\alpha_1} \text{ and } x_{\alpha_2} \leq y_{\alpha_2}$$

for all $x, y \in \mathbf{R}_L$ where $x_{\alpha_1} = \inf\{z \in \mathbf{R} \mid x(z) \geq \alpha\}$ and $x_{\alpha_2} = \sup\{z \in \mathbf{R} \mid x(z) \geq \alpha\}$ for all $x \in \mathbf{R}_L$ and for all $\alpha \in L_0$. The class

$$\{\, R_\delta|_{I_L} \;\mid\; \delta \in I \,\} \;\cup\; \{\, R^\delta|_{I_L} \;\mid\; \delta \in I \,\} \;\cup\; \{\, 0^\sim|_{I_L} \,\}$$

is a base for an L-topology $\Im$ on I_L, where R^δ and R_δ are the L-subsets of $\mathbf{R}_L$ defined by $R_\delta(x) = \bigvee\limits_{\alpha > \delta} x(\alpha)$ and $R^\delta(x) = (\bigvee\limits_{\alpha \geq \delta} x(\alpha))'$ for all $x \in \mathbf{R}_L$ and $\delta \in \mathbf{R}$ and note that $R_\delta|_{I_L}$, $R^\delta|_{I_L}$ are the restrictions of R_δ, R^δ on I_L, respectively. Recall that:

$$R^\delta(x) \wedge R^\eta(y) \leq R^{\delta+\eta}(x + y), \tag{1.3}$$

where $x+y$ is an L-real number defined by $(x+y)(\xi) = \bigvee_{\eta,\zeta\in\mathbf{R},\ \eta+\zeta=\xi} (x(\eta)\wedge y(\zeta))$ for all $\xi \in \mathbf{R}$.

2. $GT_{3\frac{1}{2}}$-spaces

Now, we shall introduce our notion of completely regular spaces in the L-case.

Definition 2.1 An L-topological space (X,τ) is said to be *completely regular* if for all $x \in X$, $F \in P(X)$ with $F \in \tau'$ and $x \notin F$, there exists an L-continuous mapping $f : (X,\tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$.

Definition 2.2 An L-topological space (X,τ) is called a *$GT_{3\frac{1}{2}}$-space* (or an *L-Tychonoff space*) if it is GT_1 and completely regular.

In the next theorem we introduce an equivalent definition for our L-completely regular spaces.

Theorem 2.1 *Let (X,τ) be an L-topological space, $\mathcal{B}$ a subbase for τ and let $\mathcal{B}'$ be the set of the complements of elements of $\mathcal{B}$. Then (X,τ) is completely regular* if and only if *for all $G \in \mathcal{B}'$ and $x \in X$, $x \notin G$ implies there exists an L-continuous mapping $f : (X,\tau) \to (I_L,\Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in G$.*

Proof. It is obvious that if the space (X,τ) is completely regular, then the condition is satisfied.

Now, let the condition be fulfilled and let $x \in X$, $F \in P(X)$ with $F \in \tau'$ and $x \notin F$. Then $x \in F' \in \tau$ and then there are $B_1,\ldots,B_n \in \mathcal{B}$ such that $x \in (B_1 \cap \cdots \cap B_n) \subseteq F'$ and then $x \in B_i$ for all $i = 1,\ldots,n$. That is, $x \notin B_i'$ for all $i = 1,\ldots,n$, and then there exists an L-continuous mapping $f_i : (X,\tau) \to (I_L,\Im)$ such that $f_i(x) = \overline{1}$ and $f_i(y) = \overline{0}$ for all $y \in B_i'$ and for all $i = 1,\ldots,n$, which is also true for all $y \in (B_1' \cup \cdots \cup B_n')$, and this means, in particular, $f_i(x) = \overline{1}$ and $f_i(y) = \overline{0}$ for all $y \in F$, for all $i = 1,\ldots,n$. Taking any one of the mappings $f_i : X \to I_L$, gives us the required L-continuous mapping $g : (X, \tau) \to (I_L, \Im)$ for which $g(x) = \overline{1}$ and $g(y) = \overline{0}$ for all $y \in F$. Thus, (X, τ) is a completely regular space. □

Now, we have an example of $GT_{3\frac{1}{2}}$-spaces.

Example 2.1 Let $X = \{x, y\}$ with $x \neq y$ and let $\tau = \{\overline{0}, \overline{1}, x_1, y_1\}$. Then $\tau' = \{\overline{0}, \overline{1}, x_1, y_1\}$ and there are only the cases of $x \notin \{y\} \in \tau'$ and $y \notin \{x\} \in \tau'$ to be studied. We shall consider the first case and the second is similar.

Since the mapping $f : (X, \tau) \to (I_L, \Im)$ defined by $f(x) = \overline{1}$ and $f(y) = \overline{0}$ is L-continuous and satisfies the condition of (X, τ) to be a completely regular space and also of being a GT_1-space, then (X, τ) is a $GT_{3\frac{1}{2}}$-space.

The following proposition and example show that the class of GT_3-spaces is larger than the class of $GT_{3\frac{1}{2}}$-spaces.

Proposition 2.1 *Every $GT_{3\frac{1}{2}}$-space is a GT_3-space.*

Proof. Let (X, τ) be $GT_{3\frac{1}{2}}$-space and let $x \notin F$ and $F \in \tau'$. That is, the space (X, τ) is GT_1 and completely regular. (X, τ) is completely regular implies there exists an L-continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$. For $R_{\frac{1}{2}}, R^{\frac{1}{2}} \in \Im$, we have $(R_{\frac{1}{2}} \circ f)(x) = R_{\frac{1}{2}}(\overline{1}) = \bigvee\limits_{\alpha > \frac{1}{2}} \overline{1}(\alpha) = 1$ and $(R^{\frac{1}{2}} \circ f)(y) = R^{\frac{1}{2}}(\overline{0}) = (\bigvee\limits_{\alpha \geq \frac{1}{2}} \overline{0}(\alpha))' = 1$ for all $y \in F$, and then f is L-continuous implies there are $h = R_{\frac{1}{2}} \circ f$, $k = R^{\frac{1}{2}} \circ f$ in L^X such that $\mathcal{N}(x)(h) \wedge \mathcal{N}(F)(k) = 1$. From that $\bigwedge\limits_{s<t} f(z)(s) \geq \bigvee\limits_{r>t} f(z)(r)$ for all $z \in X$ in general, we get for all $z \in X$ that:

$$\begin{aligned} (h \wedge k)(z) &= ((R_{\frac{1}{2}} \circ f) \wedge (R^{\frac{1}{2}} \circ f))(z) \\ &= \bigvee_{\alpha > \frac{1}{2}} f(z)(\alpha) \wedge (\bigvee_{\alpha \geq \frac{1}{2}} f(z)(\alpha))' \\ &\leq \bigwedge_{\alpha < \frac{1}{2}} f(z)(\alpha) \wedge \bigwedge_{\alpha \geq \frac{1}{2}} f(z)(\alpha)' \\ &< 1. \end{aligned}$$

Hence, $\sup(h \wedge k) < \mathcal{N}(x)(k) \wedge \bigwedge_{y \in F} \mathcal{N}(y)(h)$ and therefore (X, τ) is a regular space and consequently it is a GT_3-space. □

In this example we introduce a GT_3-space which is not $GT_{3\frac{1}{2}}$-space.

Example 2.2 Let $X = \{x, y\}$ with $x \neq y$ and let $\tau = \{\overline{0}, \overline{1}, y_{\frac{1}{2}}, y_1, x_{\frac{3}{4}} \vee y_{\frac{1}{2}}, x_{\frac{3}{4}} \vee y_1\}$. Then $\tau' = \{\overline{0}, \overline{1}, x_{\frac{1}{4}}, x_1, x_{\frac{1}{4}} \vee y_{\frac{1}{2}}, x_1 \vee y_{\frac{1}{2}}\}$ and there is only the case of $y \notin \{x\} \in \tau'$ to be studied. Since $f = x_{\frac{3}{4}} \vee y_{\frac{1}{2}}$ and $g = y_1$ in L^X implies

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) = \operatorname{int}_\tau f(x) \wedge \operatorname{int}_\tau g(y) = \frac{3}{4} > \frac{1}{2} = \sup(f \wedge g),$$

then $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist and hence (X, τ) is a regular space and it is also a GT_1-space. Thus (X, τ) is a GT_3-space.

Since in case of $y \notin \{x\} \in \tau'$ we get that any mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(y) = \overline{1}$ and $f(x) = \overline{0}$ is not L-continuous, then (X, τ) is not completely regular and thus it is not a $GT_{3\frac{1}{2}}$-space.

L-topogenous orders. A binary relation $\ll$ on L^X is said to be an *L-topogenous order* on X ([21]) if the following conditions are fulfilled:

(1) $\overline{0} \ll \overline{0}$ and $\overline{1} \ll \overline{1}$,

(2) $f \ll g$ implies $f \leq g$,

(3) $f_1 \leq f \ll g \leq g_1$ implies $f_1 \ll g_1$ and

(4) from $f_1 \ll g_1$ and $f_2 \ll g_2$ it follows $f_1 \vee f_2 \ll g_1 \vee g_2$ and $f_1 \wedge f_2 \ll g_1 \wedge g_2$.

An L-topogenous order $\ll$ is said to be *regular* or is said to be an *L-topogenous structure* if for all $f, g \in L^X$ with $f \ll g$ there is a $k \in L^X$ such that $f \ll k$ and $k \ll g$ hold.

An L-topogenous structure $\ll$ is called *complementarily symmetric* if $f \ll g$ implies $g' \ll f'$ for all $f, g \in L^X$.

Let $X \neq \emptyset$ be an arbitrary set. By an *L-function family* Φ on X, we mean the set of all L-real functions $f : X \to I_L$.

Let f and g be L-sets in X. Then a function $h : X \to I_L$ is said to *separate* f and g if $\overline{0} \leq h(x) \leq \overline{1}$ for all $x \in X$, $x_1 \leq f$ implies $h(x) = \overline{1}$ and $y_1 \leq g$ implies $h(y) = \overline{0}$. Moreover, if Φ is an L-function family on X, then the sets $f, g \in L^X$ are called Φ-*separated* or Φ-*separable* if there exists a function $h \in \Phi$ separating them.

L-proximities. A binary relation δ on L^X is called an *L-L-proximity* (or an *L-proximity*) ([13, 19]) on X provided it fulfills the following conditions:

(P1) $f \,\overline{\delta}\, g$ implies $g \,\overline{\delta}\, f$, where $\overline{\delta}$ means $(L^X \times L^X) \setminus \delta$, called the *negation* of δ.

(P2) $(f \vee g) \,\overline{\delta}\, h$ if and only if $f \,\overline{\delta}\, h$ and $g \,\overline{\delta}\, h$.

(P3) $f = \overline{0}$ or $g = \overline{0}$ implies $f \,\overline{\delta}\, g$ for all $f, g \in L^X$.

(P4) $f \,\overline{\delta}\, g$ implies $f \leq g'$.

(P5) If $f \,\overline{\delta}\, g$, then there is an $h \in L^X$ such that $f \,\overline{\delta}\, h$ and $h' \,\overline{\delta}\, g$.

(X, δ) is called an *L-proximity space*. The L-proximity δ on a set X is associated with an L-topology τ_δ. The related interior and closure operators int_δ and cl_δ are given by:

$$\mathrm{int}_\delta f = \bigvee_{f' \,\overline{\delta}\, g} g, \quad \text{and} \quad \mathrm{cl}_\delta f = \bigwedge_{g' \,\overline{\delta}\, f} g$$

respectively, for all $f \in L^X$. An L-set f in L^X is called a τ_δ-*neighborhood* of $x \in X$ if $x_1 \,\overline{\delta}\, f'$. A mapping f between L-proximity spaces (X, δ) and (Y, ρ) is called *L-proximally continuous* (or (δ, ρ)-*continuous*) provided $g \,\overline{\rho}\, h$ implies $(g \circ f) \,\overline{\delta}\, (h \circ f)$ for all $g, h \in L^Y$.

Proposition 2.2 [13, 21] *There is an identification between the L- proximity δ on X and the complementarily symmetric L-topogenous structure $\ll$ on X given by*

$$f \ll g' \Leftrightarrow f \,\overline{\delta}\, g \tag{2.1}$$

for all $f, g \in L^X$.

Let $(\ll_n)$ be a sequence of L-topogenous structures on X and $(\prec_n)$ a sequence of L-topogenous structures on I_L. Then an L-real function $f : X \to I_L$ is said to be *associated with* the sequence $(\ll_n)$ if for all $g, h \in L^{I_L}$, $g \prec_n h$ implies $(g \circ f) \ll_{n+1} (h \circ f)$ for every positive integer n.

Remark 2.1 Consider $(\ll_n)$ and $(\prec_n)$ are two sequences of two complementarily symmetric L-topogenous structures $\ll$ and $\prec$ on X and I_L, respectively. Let δ and δ^* be the L-proximities on X and I_L identified with $\ll$ and $\prec$ by (2.1), respectively. Then for a function $f : X \to I_L$ associated with the sequence $\ll$, we get from (2.1) that $g\, \overline{\delta^*}\, h$ implies $(g \circ f)\, \overline{\delta}\, (h \circ f)$ for all $g, h \in L^{I_L}$, which means that f is L-proximally continuous.

Here, to prove the Urysohn's Lemma for our notion of $GT_{3\frac{1}{2}}$-spaces, we need the following results.

In the proof of the following lemma we use the way of Császár ([8]).

Lemma 2.1 *Suppose that $\ll_n$ $(n = 0, 1, 2, \ldots)$ are complementarily symmetric L-topogenous structures on a set X. If $F, G \in P(X)$ and $\chi_F \ll_0 \chi_G$, then there exists a function $f : X \to I_L$ associated with the sequence $(\ll_n)$ for which $f(x) = \overline{0}$ for all $x \in F$ and $f(y) = \overline{1}$ for all $y \in G'$.*

Proof. Since $(\ll_n)$ is a sequence of binary relations in the crisp case and fulfill the conditions of being complementarily symmetric L-topogenous structures, then we can deduce that there is a recursion process in the crisp case similar to that in the usual case in [8] by defining the order relation $\ll_m$ for $m \in R$ where R denotes the set of all non-negative dyadic rational numbers $(m = \frac{p}{2^n}; p = 0, \ldots, 2^n, n = 0, 1, \ldots)$. With this relation, the sets $A(m)$ can be associated such that

(1) $A(0) = F$, $A(1) = G$;

(2) $A(\frac{p}{2^n}) \ll_m A(\frac{p+1}{2^n})$, $A(r) = X$ for all $r > 1$; $p = 0, \ldots, 2^n, n = 0, 1, \ldots$.

From the properties of $\ll_n$ we get $A(r) \subseteq A(s)$ for all $r, s \in R$, $r < s$.

Define the L-real function $f : X \to I_L$ by $f(x) = \bigwedge_{r \in R} \{\overline{r} \mid x \in A(r)\}$, then $f(x) = \overline{0}$ for all $x \in F$ and $f(y) = \overline{1}$ for all $y \in G'$.

Now, as in the usual case, f itself is an associated function with the sequence $(\ll_n)$ and also f separates the sets χ_F and $\chi_{G'}$. Hence the proof is complete. □

Proposition 2.3 [13] *If $f : (X, \delta) \to (Y, \rho)$ is L-proximally continuous, then $f : (X, \tau_\delta) \to (Y, \tau_\rho)$ is (τ_δ, τ_ρ)-continuous.*

Proposition 2.4 [6] *If (X, τ) is a normal L-topological space, then the binary relation δ on L^X defined by*

$$f \,\overline{\delta}\, g \iff \mathcal{N}(\mathrm{cl}_\tau f) \leq (\dot{\mathrm{cl}_\tau} g)' \tag{2.2}$$

is an L-proximity on X. Conversely, in an L-proximity space (X, δ) with δ fulfills (2.2)*, the L-topological space (X, τ_δ) is normal.*

From (2.1), Lemma 2.1 and Remark 2.1 we can easily deduce the following.

Proposition 2.5 *Let F and G be subsets of X with $\chi_F \,\overline{\delta}\, \chi_G$ in the L-proximity space (X, δ) and let Φ be the family of those L-proximally continuous functions h of (X, δ) into the L-proximity space (I_L, δ^*) for which $x \in X$ implies $\overline{0} \leq h(x) \leq \overline{1}$. Then χ_F and χ_G are Φ-separable.*

Proof. Let $\ll$ be the complementarily symmetric L-topogenous structure identified with δ. From (2.1), $\chi_F \,\overline{\delta}\, \chi_G$ implies that $\chi_F \ll \chi'_G$ and since $h \in \Phi$ is L-proximally continuous, then h, by means of Remark 2.1, is associated with $\ll$. Hence, Lemma 2.1 implies that χ_F and χ_G are separated by h, that is, χ_F and χ_G are Φ-separable. □

We shall use the following results in the proof of the next proposition.

Lemma 2.2 [6] *For all $f, g \in L^X$, we have*

$$f \leq g \;\text{ if and only if }\; \dot{f} \leq \dot{g}.$$

Proposition 2.6 [3] *The L-topological space (X, τ) is GT_1* if and only if $\mathrm{cl}\,\dot{x} = \dot{x}$, *that is,* $\mathrm{cl}_\tau\{x\} = \{x\}$, *that is,* $\mathrm{cl}_\tau x_1 = x_1$ *for all $x \in X$.*

The L-proximity induced in Proposition 2.4 is also compatible with the GT_4-topologies. We shall prove now the following important result.

Proposition 2.7 *Let (X, τ) be a normal L-topological space and δ the L-proximity on X defined by* (2.2). *Then τ_δ is coarser than τ. The equality $\tau = \tau_\delta$ holds* if and only if *the space (X, τ) is a GT_4-space.*

Proof. Let (X, τ) be a normal space. If f is a τ_δ-neighborhood of x, then $x_1 \,\overline{\delta}\, f'$. Hence, from (2.2) we have $\mathcal{N}(\mathrm{cl}_\tau x_1) \leq (\mathrm{cl}_\tau \dot{f}')'$, and therefore

$$\dot{x} \leq \mathcal{N}(x) \leq \mathcal{N}(\mathrm{cl}_\tau\{x\}) = \mathcal{N}(\mathrm{cl}_\tau x_1) \leq (\mathrm{cl}_\tau \dot{f}')' \leq \dot{f}.$$

From Lemma 2.2 we get that $x_1 \leq (\mathrm{cl}_\tau f')' \leq f$ and $(\mathrm{cl}_\tau f')' \in \tau$. Hence, f is a τ-neighborhood of x. Thus τ_δ is coarser than τ, that is, $\tau_\delta \subseteq \tau$.

Now, let (X, τ) be a GT_4-space and let $\mathcal{N}_\tau(x)$ and $\mathcal{N}_{\tau_\delta}(x)$ denote for the L-neighborhood filters at x of the spaces (X, τ) and (X, τ_δ), respectively. Then (X, τ) is normal and a GT_1-space and hence $\tau_\delta \subseteq \tau$ and $\mathcal{N}_\tau(x) \not\leq \dot{y}$ for all $y \neq x$ in X. From that $\tau_\delta \subseteq \tau$ implies $\mathcal{N}_\tau(x) \leq \mathcal{N}_{\tau_\delta}(x)$ for all x in X, we get that $\mathcal{N}_{\tau_\delta}(x) \geq \mathcal{N}_\tau(x) \not\leq \dot{y}$ for all $y \neq x$ in X. Hence from (1.1), we have $\mathcal{N}_{\tau_\delta}(x) \not\leq \dot{y}$ for all $y \neq x$ in X and thus (X, τ_δ) is also a GT_1-space. Thus from Proposition 2.6 we get that $x_1 \in \tau_\delta'$ for all $x \in X$. If f is a τ-neighborhood of x, then $f' \leq x_1'$ and since $x_1' \in \tau_\delta$, then x_1' is a τ_δ-neighborhood of every $y \in X$ with $y_1 \leq f'$. This means that $f' \,\overline{\delta}\, x_1$ which implies that f is a τ_δ-neighborhood of x. Hence, $\tau \subseteq \tau_\delta$ and thus (X, τ) is a GT_4-space implies that $\tau = \tau_\delta$.

Conversely; let $\tau = \tau_\delta$ and $x \in X$, and let f be a τ-neighborhood of x. Then $f \in \tau_\delta$ and $x_1 \leq f$, which means that

$$(\dot{\mathrm{cl}_\tau} x_1) \leq \mathcal{N}(\mathrm{cl}_\tau x_1) \leq (\dot{\mathrm{cl}_\tau} f')' \leq \dot{f},$$

and this means from Lemma 2.2 that $\mathrm{cl}_\tau x_1 \leq f$. Hence, $x_1 \leq f$ implies $\mathrm{cl}_\tau x_1 \leq f$, and then $\mathrm{cl}_\tau x_1 \leq x_1$ for all $x \in X$. Thus, $\mathrm{cl}_\tau x_1 = x_1$ for all $x \in X$ and (X, τ) is a GT_1-space. Since $(X, \tau = \tau_\delta)$, by means of Proposition 2.4, is a normal space, then (X, τ) is a GT_4-space. □

In a strictly speaking we can easily find out the Urysohn's Lemma as follows.

Lemma 2.3 (Urysohn's Lemma) *Let (X, τ) be an L-topological space. Then (X, τ) is normal* if and only if *for all $F, G \in P(X)$ with F, G are disjoint closed sets in X, there exists an L-continuous function $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{0}$ for all $x \in F$ and $f(y) = \overline{1}$ for all $y \in G$.*

Proof. Let (X, τ) be a normal L-topological space. Then the infimum $\mathcal{N}(F) \wedge \mathcal{N}(G)$ does not exist for all $F, G \in P(X)$ with F, G are disjoint closed sets in X and hence $\mathcal{N}(F) \leq \dot{G}'$. Thus if δ is the L-proximity on X defined by (2.2), then we have $\chi_F \, \overline{\delta} \, \chi_G$. By Proposition 2.5 an L-proximally continuous function $f : (X, \delta) \to (I_L, \delta^*)$ exists and separates F and G, where δ^* is an L-proximity on I_L. From Proposition 2.3, we get that f is a $(\tau_\delta, \Im_{\delta^*})$-continuous function, and from Proposition 2.7, we have $\tau_\delta \subseteq \tau$ and then f is a $(\tau, \Im)$-continuous function and $f(x) = \overline{0}$ for all $x \in F$ and $f(y) = \overline{1}$ for all $y \in G$.

Conversely; If there exists an L-continuous function $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{0}$ for all $x \in F$ and $f(y) = \overline{1}$ for all $y \in G$ for all $F, G \in P(X)$ with F, G are disjoint closed sets in X and then, when we consider the sets $R_{\frac{1}{2}}$ and $R^{\frac{1}{2}}$ are restricted on I_L, we get that the two L-sets $g = R^{\frac{1}{2}} \circ f$ and $h = R_{\frac{1}{2}} \circ f$ are open L-sets such that

$$\mathcal{N}(F)(g) = \bigwedge_{x \in F} g(x) = \bigwedge_{x \in F} R^{\frac{1}{2}}(f(x)) = \bigwedge_{x \in F} (\bigvee_{\alpha \geq \frac{1}{2}} f(x)(\alpha))' = 1$$

and

$$\mathcal{N}(G)(h) = \bigwedge_{y\in G} h(y) = \bigwedge_{y\in G} R_{\frac{1}{2}}(f(y)) = \bigwedge_{y\in G} \bigvee_{\alpha>\frac{1}{2}} f(y)(\alpha) = 1.$$

That is, $\mathcal{N}(F)(g) \wedge \mathcal{N}(G)(h) = 1$. Since

$$(g\wedge h)(z) = \bigvee_{\alpha>\frac{1}{2}} f(z)(\alpha) \wedge (\bigvee_{\alpha\geq\frac{1}{2}} f(x)(\alpha))' \leq \bigwedge_{\alpha<\frac{1}{2}} f(z)(\alpha) \wedge \bigwedge_{\alpha\geq\frac{1}{2}} f(x)(\alpha)' < 1$$

for all $z \in X$, then $\mathcal{N}(F) \wedge \mathcal{N}(G)$ does not exist and therefore (X, τ) is a normal space. □

To prove that the $GT_{3\frac{1}{2}}$-spaces are more general than the GT_4-spaces we need the following result.

Proposition 2.8 *Every GT_4-space is a $GT_{3\frac{1}{2}}$-space.*

Proof. Let (X, τ) be a GT_4-space and let $x \notin F$ and $F \in \tau'$. Then (X, τ) is a GT_1-space and hence from Proposition 2.6, we have $\{x\}, F$ are disjoint closed subsets of X and then, by Urysohn's Lemma, there exists an L-continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$. Hence, (X, τ) is a $GT_{3\frac{1}{2}}$-space. □

We have the following example for a $GT_{3\frac{1}{2}}$-space which is not a GT_4-space.

Example 2.3 Let (X, τ) be the L-topological space defined as the Moore Space. That is, X is the closed upper half plane $\{(x, y) \mid y \geq 0\}$ in R^2 and τ is defined as follows: For each point in the open upper half plane, $\{(x, y) \mid y > 0\}$, the basic L-neighborhoods will be the usual open disks, and at the points z on the X-axis, the basic L-neighborhoods will be the sets $\{z\} \cup A$, where A is an open disk in the open upper half plane and tangent to the x-axis at z.

As in the classical case, since there are two disjoint closed sets $A = \{(r, 0) \mid r \in Q\}$ and $B = \{(s, 0) \mid s \in Q'\}$ for which they have no disjoint L-neighborhoods, where Q and Q' denote for the rational and the irrational numbers, respectively, then we get that (X, τ) is GT_1 and it is not normal. That is, (X, τ) is not a GT_4-space.

Now, let $p \in X$ and V a basic open neighborhood of p (so that V is either an open disk centered at p or else p together with an open disk tangent to p, depending on the placement of p). Define $f : (X, \tau) \to (I_L, \Im)$ by $f(p) = \overline{1}$ and $f(x) = \overline{0}$ for all $x \notin V$, and defining f linearly along the straight line segments between p and the points on the boundary of V. Then f is an L-continuous mapping on X such that $f(p) = \overline{1}$ and $f(x) = \overline{0}$ for all $x \in V'$. Since any closed set in X which does not contain p is contained in U' for some basic L-neighborhood U of p, it follows that (X, τ) is completely regular and thus it is a $GT_{3\frac{1}{2}}$-space.

L-metric spaces. In the sequel will be shown that the L-metric space in sense of S. Gähler and W. Gähler, which had been introduced in [10], is an example of our $GT_{3\frac{1}{2}}$-space. By an *L-metric* on a set X we mean ([10]) a mapping $\varrho : X \times X \longrightarrow \mathbf{R}_L^*$ such that the following conditions are fulfilled:

(1) $\varrho(x, y) = 0^\sim$ if and only if $x = y$

(2) $\varrho(x, y) = \varrho(y, x)$ (symmetry)

(3) $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$ (triangle inequality).

A set X equipped with an L-metric ϱ on X is called an *L-metric space*.

Note that $0^\sim$ denotes the L-number which has values 1 at 0 and 0 otherwise.

To each L-metric ϱ on a set X is generated canonically a stratified L-topology τ_ϱ which has $\{\varepsilon \circ \varrho_x \mid \varepsilon \in \mathcal{E},\ x \in X\}$ as a base, where $\varrho_x : X \to \mathbf{R}_L^*$ is the mapping defined by $\varrho_x(y) = \varrho(x, y)$ and

$$\mathcal{E} \ = \ \{\, \overline{\alpha} \wedge R^\delta|_{\mathbf{R}_L^*} \ \mid \ \delta > 0,\ \alpha \in L \,\} \cup \{\, \overline{\alpha} \ \mid \ \alpha \in L \,\},$$

here $\overline{\alpha}$ has $\mathbf{R}_L^*$ as domain and $R^\delta|_{\mathbf{R}_L^*}$ is the restriction of R^δ on $\mathbf{R}_L^*$.

In the following proposition we shall prove that every L-metric space in sense of S. Gähler and W. Gähler ([10]) is a GT_4-space.

Proposition 2.9 *Any L-metric space (X, τ_ϱ) is a GT_4-space.*

Proof. Let F and G be two disjoint closed subsets of (X, τ_ϱ). Then for all $x \in F$ and $y \in G$ we get $\varrho(x, y) \neq \widetilde{0}$, that is, there exists $\delta > 0$ such that $\varrho(x, y)(2\delta) > 0$ and then $R^{2\delta}|_{\mathbf{R}_L^*}(\varrho(x, y)) = (\bigvee\limits_{\alpha \geq 2\delta} \varrho(x, y)(\alpha))' < 1$.

Let $f = R^\delta|_{\mathbf{R}_L^*} \circ \varrho_x$ and $g = R^\delta|_{\mathbf{R}_L^*} \circ \varrho_y$. Then

$$f(x) = R^\delta|_{\mathbf{R}_L^*}(\varrho_x(x)) = R^\delta|_{\mathbf{R}_L^*}(\widetilde{0}) = (\bigvee_{\alpha \geq \delta} \widetilde{0}(\alpha))' = 1 \text{ for all } x \in F,$$

and

$$g(y) = R^\delta|_{\mathbf{R}_L^*}(\varrho_y(y)) = R^\delta|_{\mathbf{R}_L^*}(\widetilde{0}) = (\bigvee_{\alpha \geq \delta} \widetilde{0}(\alpha))' = 1 \text{ for all } y \in G.$$

That is, f and g are open L-neighborhoods in τ_ϱ at all $x \in F$ and all $y \in G$, respectively, which means $\bigwedge\limits_{x \in F} \mathcal{N}(x)(f) \wedge \bigwedge\limits_{y \in G} \mathcal{N}(y)(g) = 1$. From the symmetry and the triangle inequality of ϱ and from (1.3) we get $R^\delta|_{\mathbf{R}_L^*}(\varrho(x, z)) \wedge R^\delta|_{\mathbf{R}_L^*}(\varrho(y, z)) \leq R^{2\delta}|_{\mathbf{R}_L^*}(\varrho(x, y)) < 1$ and hence $(f \wedge g)(z) = (R^\delta|_{\mathbf{R}_L^*} \circ \varrho_x)(z) \wedge (R^\delta|_{\mathbf{R}_L^*} \circ \varrho_y)(z) < 1$ for all $z \in X$ and hence $\sup(f \wedge g) < 1$. Therefore, the space (X, τ_ϱ) is normal and it is clear that (X, τ_ϱ) is a GT_1-space. Thus, (X, τ_ϱ) is a GT_4-space. $\square$

Example 2.4 From Propositions 2.8 and 2.9 we get that the L-metric space in sense of S. Gähler and W. Gähler [10] is an example of our notion of $GT_{3\frac{1}{2}}$-space and thus it is also an example of all our GT_i-spaces, $i = 0, 1, 2, 3, 4$.

A topological space (X, T) is called T_1 if for any $x \neq y$ in X, there exist neighborhoods $\mathcal{O}_x$ of x such that $y \notin \mathcal{O}_x$ and $\mathcal{O}_y$ of y such that $x \notin \mathcal{O}_y$. A topological space (X, T) is called *completely regular* if for all $x \notin F \in T'$, there exists a continuous mapping $f : (X, T) \to (I, T_I)$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in F$, where T_I is the usual topology on the closed unit interval I. A topological space (X, T) is called $T_{3\frac{1}{2}}$ (or *Tychonoff*) if it is T_1 and completely regular ([8]).

To show that our notion of $GT_{3\frac{1}{2}}$-space is an extension with respect to the functor ω in sense of Lowen ([23]), we need the following proposition.

Proposition 2.10 [2] *A topological space* (X,T) *is* T_1 if and only if *the induced* L*-topological space* $(X,\omega(T))$ *is* GT_1.

Proposition 2.11 *A topological space* (X,T) *is* $T_{3\frac{1}{2}}$ if and only if *the induced* L*-topological space* $(X,\omega(T))$ *is* $GT_{3\frac{1}{2}}$.

Proof. By means of Proposition 2.10, we have (X,T) is a T_1-space if and only if $(X,\omega(T))$ is a GT_1-space.

Now, let $x \notin F$ and $F \in \omega(T)'$ hold. Then, $(s_\alpha F)' = s_\alpha F' \in T$ and hence $x \notin s_\alpha F \in T'$. Since (X,T) is a completely regular space, then there exists a continuous mapping $g : (X,T) \to (I,T_I)$ such that $g(x) = 1$, $g(y) = 0$ for all $y \in s_\alpha F = F$ for all $\alpha \in L_1$. It is obvious that $g : (X,\omega(T)) \to (I,\omega(T_I))$ is also L-continuous. Take $h : (I,\omega(T_I)) \to (I_L,\Im)$ defined by $h(z) = \overline{z}$ for all $z \in I$ which is L-continuous. Then $f = h \circ g : (X,\omega(T)) \to (I_L,\Im)$ is L-continuous and $f(x) = \overline{1}$, $f(y) = \overline{0}$ for all $y \in F$. Thus, $(X,\omega(T))$ is a completely regular space and therefore $(X,\omega(T))$ is a $GT_{3\frac{1}{2}}$-space.

Conversely; if $(X,\omega(T))$ is a completely regular space and $x \notin F$, $F \in T'$, then $x \notin \chi_F \in \omega(T)'$ which means that there exists an L-continuous mapping $f : (X,\omega(T)) \to (I_L,\Im)$ and $f(x) = \overline{1}$, $f(y) = \overline{0}$ for all $y \in F$. Then from that $T = (\omega(T))_\alpha$ and $\Im_\alpha = T_I$ ([18]), there could be found a mapping $f_\alpha : (X,T) \to (I,T_I)$ which is continuous and $f_\alpha(x) = 1$, $f_\alpha(y) = 0$ for all $y \in F$, for all $\alpha \in L_1$. Hence, (X,T) is a completely regular space and then (X,T) is a $T_{3\frac{1}{2}}$-space. □

We shall use the following result.

Proposition 2.12 [2] *Let* (X,τ) *be a* GT_1*-space and let* σ *be an* L*-topology on* X *finer than* τ*. Then* (X,σ) *also is a* GT_1*-space.*

The following proposition shows that the finer L-topological space of a $GT_{3\frac{1}{2}}$-space also is a $GT_{3\frac{1}{2}}$-space.

Proposition 2.13 *Let (X, τ) be a $GT_{3\frac{1}{2}}$-space and let σ be an L-topology on X finer than τ. Then (X, σ) also is a $GT_{3\frac{1}{2}}$-space.*

Proof. From Proposition 2.12, we get (X, σ) is a GT_1-space. Consider a subbase $\mathcal{B}$ for τ and let $x \in X$, $F \in \sigma'$ with $x \notin F$. Then there are $B_1, \ldots, B_n \in \mathcal{B}$ such that $x \in (B_1 \cap \cdots \cap B_n) \subseteq F'$. So, $x \notin B_i'$, $B_i' \in \tau'$ for all $i = 1, \ldots, n$. From Theorem 2.1, we get that there exists an L-continuous mapping $f_i : (X, \tau) \to (I_L, \Im)$ such that $f_i(x) = \overline{1}$ and $f_i(y) = \overline{0}$ for all $y \in B_i'$ and for all $i = 1, \ldots, n$, which is also true for all $y \in (B_1' \cup \cdots \cup B_n')$, and this means, in particular, that $f_i(x) = \overline{1}$ and $f_i(y) = \overline{0}$ for all $y \in F$; for all $i = 1, \ldots, n$. Since $\tau \subseteq \sigma$, then anyone of these mappings $f_i : X \to I_L$, gives us the required L-continuous mapping $g : (X, \sigma) \to (I_L, \Im)$ for which $g(x) = \overline{1}$ and $g(y) = \overline{0}$ for all $y \in F$. Thus, (X, σ) is a completely regular space, and therefore it is a $GT_{3\frac{1}{2}}$-space. □

3. Initial $GT_{3\frac{1}{2}}$-spaces

Denote by L-**Tych** the category of all $GT_{3\frac{1}{2}}$-spaces (or L-Tychonoff spaces). L-**Tych** is a concrete category. We may note that the category L-**Tych** is a full subcategory of the category L-**Top** of L-topological spaces, which is topological over the category **Set**, and hence all initial lifts exist uniquely in the category L-**Tych** and this means it is topological over **Set**. That is, all initial $GT_{3\frac{1}{2}}$-spaces exist in L-**Tych**.

Proposition 1.1 states that $\bigvee\limits_{i \in I} f_i^{-1}(\tau_i)$ is the initial L-topology of a family $(\tau_i)_{i \in I}$ of topologies with respect to a family $(f_i)_{i \in I}$ of mappings. The initial L-topology of a family of $GT_{3\frac{1}{2}}$-topologies is precisely the initial $GT_{3\frac{1}{2}}$-topology, this result will be shown in the following propositions.

At first consider the case of one mapping.

Proposition 3.1 *Let $f : X \to Y$ be an injective mapping and (Y, σ) a $GT_{3\frac{1}{2}}$-space. Then the initial L-topological space $(X, \tau = f^{-1}(\sigma))$, of (Y, σ) with respect to f, also* *is* $GT_{3\frac{1}{2}}$*-space.*

Proof. From (2) in Proposition 1.3 we have that (X, τ) is a GT_1-space.

Now, let $x \notin F$ and $F \in \tau' = (f^{-1}(\sigma))' = f^{-1}(\sigma')$. From that f is injective it follows $f(x) \notin f(F)$ and $f(F) \in \sigma'$. Since (Y, σ) is $GT_{3\frac{1}{2}}$-space, then there exists an L-continuous mapping $g : (Y, \sigma) \to (I_L, \Im)$ such that $g(f(x)) = \overline{1}$ and $g(f(y)) = \overline{0}$ for all $y \in F$. Then $g \circ f : (X, \tau) \to (I_L, \Im)$ is L-continuous and thus (X, τ) is completely regular. Therefore, (X, τ) is a $GT_{3\frac{1}{2}}$-space. □

Assume now that a family $((X_i, \tau_i))_{i\in I}$ of $GT_{3\frac{1}{2}}$-spaces and a family $(f_i)_{i\in I}$ of injective mappings $f_i : X \to X_i$ for some $i \in I$ are given, where I may be any class.

Proposition 3.2 *The initial L-topological space* $(X, \tau = \bigvee_{i\in I} f_i^{-1}(\tau_i))$, *of the family* $((X_i, \tau_i))_{i\in I}$ *with respect* $(f_i)_{i\in I}$, *also is* $GT_{3\frac{1}{2}}$*-space.*

Proof. Similarly, as in proof of Proposition 3.1. □

$GT_{3\frac{1}{2}}$-subspaces and $GT_{3\frac{1}{2}}$-product spaces. The $GT_{3\frac{1}{2}}$-subspaces and the $GT_{3\frac{1}{2}}$-product spaces, in the categorical sense, are special initial $GT_{3\frac{1}{2}}$-spaces ([1]) and therefore these spaces can be characterized as follows: Let (X, τ) be $GT_{3\frac{1}{2}}$-space and A a non-empty subset of X and $i : A \hookrightarrow X$ the inclusion mapping, and let (A, τ_A) be the subspace of (X, τ), that is, τ_A be the initial of τ with respect to i. Then, from Proposition 3.1, (A, τ_A) is a $GT_{3\frac{1}{2}}$-space. Let X be the cartesian product $\prod_{i\in I} X_i$ of the family $(X_i)_{i\in I}$ and $p_i : X \to X_i$ be the related projections, and let for each $i \in I$, (X_i, τ_i) be a $GT_{3\frac{1}{2}}$-space. Let $(X, \prod_{i\in I} \tau_i)$ be the product space of $((X_i, \tau_i))_{i\in I}$. That is, $\prod_{i\in I} \tau_i$ is the initial of $(\tau_i)_{i\in I}$ with respect to $(p_i)_{i\in I}$ and hence, by means of Proposition 3.2, we get that $(X, \prod_{i\in I} \tau_i)$ is a $GT_{3\frac{1}{2}}$-space.

We summarize the last dissection in the following corollary.

Corollary 3.1 *The L-topological subspaces and the L-topological product spaces of a family of* $GT_{3\frac{1}{2}}$*-spaces also are* $GT_{3\frac{1}{2}}$*-spaces.*

4. Final $GT_{3\frac{1}{2}}$-spaces

Since the category L-**Tych** is topological, then all final $GT_{3\frac{1}{2}}$-spaces also exist ([1]).

By means of Proposition 1.1, $\bigwedge_{i\in I} f_i(\tau_i)$ is the final L-topology of a family $(\tau_i)_{i\in I}$ of topologies with respect to a family $(f_i)_{i\in I}$ of mappings. The following propositions show that the final L-topology of a family of $GT_{3\frac{1}{2}}$-topologies is a $GT_{3\frac{1}{2}}$-topology.

In case of one mapping we get this result.

Proposition 4.1 *Let $f : X \to Y$ be a surjective L-open mapping and (X, τ) a $GT_{3\frac{1}{2}}$-space. Then the final L-topological space $(Y, \sigma = f(\tau))$, of (X, τ) with respect to f, also is $GT_{3\frac{1}{2}}$-space.*

Proof. Let $y \notin H$ and $H \in (f(\tau))' \subseteq f(\tau')$ hold. From that f is surjective, it follows that there exists $x \notin F$, $F \in \tau'$, where $x = f^{-1}(y)$ and $F = f^{-1}(H)$. Since (X, τ) is a $GT_{3\frac{1}{2}}$-space, then there exists an L-continuous mapping $g : (X, \tau) \to (I_L, \Im)$ such that $g(x) = \overline{1}$ and $g(z) = \overline{0}$ for all $z \in F$. That is, $g(f^{-1}(y)) = \overline{1}$ and $g(f^{-1}(s)) = \overline{0}$ for all $s \in H$, which means that there exists a mapping $h = (g \circ f^{-1}) : (Y, \sigma) \to (I_L, \Im)$ such that $h(y) = \overline{1}$ and $h(s) = \overline{0}$ for all $s \in H$. Since f is L-open, then $\text{int}_\tau \lambda \circ f^{-1} = f(\text{int}_\tau \lambda) \leq \text{int}_{f(\tau)}(f(\lambda)) = \text{int}_{f(\tau)}(\lambda \circ f^{-1})$ for all $\lambda \in L^X$ which means that $f^{-1} : (Y, \sigma) \to (X, \tau)$ is L-continuous and hence the composition $h = g \circ f^{-1} : (Y, \sigma) \to (I_L, \Im)$ is L-continuous and the space (Y, σ) is a completely regular space. From (2) in Proposition 1.3 we have (Y, σ) is a GT_1-space and hence (Y, σ) is the finest $GT_{3\frac{1}{2}}$-space. □

For any class I we have the following result.

Proposition 4.2 *Let $((X_i, \tau_i))_{i\in I}$ be a family of $GT_{3\frac{1}{2}}$-spaces and $(f_i)_{i\in I}$ a family of mappings $f_i : X_i \to X$ which are surjective L-open for some $i \in I$. Then the final L-topological space $(X, \tau = \bigwedge_{i\in I} f_i(\tau_i))$, of the family $((X_i, \tau_i))_{i\in I}$ with respect $(f_i)_{i\in I}$, also is $GT_{3\frac{1}{2}}$-space.*

Proof. By a similar way as in proof of Proposition 4.1. □

The final L-topology can be constructed by the initial L-topology as follows ([1]):

Remark 4.1 For a family $((X_i, \tau_i))_{i\in I}$ of L-topological spaces and a family $(f_i)_{i\in I}$ of mappings f_i of sets X_i into a set X, the final L-topology τ of $(\tau_i)_{i\in I}$ with respect $(f_i)_{i\in I}$ is the infimum of the set S of all L-topologies σ on X for which each τ_i, $i \in I$, is finer than the initial L-topology of σ with respect to f_i. That is, $\tau = \bigwedge\limits_{\sigma\in S} \sigma$. It is clear that the L-topologies introduced in Propositions 4.1 and 4.2 coincide with this final L-topology $\tau = \bigwedge\limits_{\sigma\in S} \sigma$.

$GT_{3\frac{1}{2}}$-quotient spaces and $GT_{3\frac{1}{2}}$-sum spaces. The $GT_{3\frac{1}{2}}$-quotient spaces and the $GT_{3\frac{1}{2}}$-sum spaces, in the categorical sense, are special final $GT_{3\frac{1}{2}}$-spaces ([1]) and therefore these spaces can be characterized as follows: Let (X, τ) be a $GT_{3\frac{1}{2}}$-space and $f : X \to Y$ a surjective mapping. Then the quotient space $(Y, f(\tau))$, by means of Proposition 4.1, is a $GT_{3\frac{1}{2}}$-space. Let $((X_i, \tau_i))_{i\in I}$ be a family of $GT_{3\frac{1}{2}}$-spaces and let $(X, \bigoplus\limits_{i\in I} \tau_i)$ be the L-topological sum space of the family $((X_i, \tau_i))_{i\in I}$. That is, $\bigoplus\limits_{i\in I} \tau_i$ is the final of $(\tau_i)_{i\in I}$ with respect to $(e_i)_{i\in I}$. Hence Proposition 4.2 implies that $(X, \bigoplus\limits_{i\in I} \tau_i)$ is a $GT_{3\frac{1}{2}}$-space.

The last dissection can be summarized in the following corollary.

Corollary 4.1 *The L-topological quotient spaces and the L-topological sum spaces of a family of $GT_{3\frac{1}{2}}$-spaces also are $GT_{3\frac{1}{2}}$-spaces.*

5. The relation between our completely regular spaces and other notions of completely regular spaces

In this section is shown that our completely regular spaces are more general than the completely regular spaces defined by Hutton in [16], by Katsaras in [20] and by Kandil and El-Shafee in [17].

The relation between our completely regular spaces and the completely regular spaces in sense of Hutton. In [16] Hutton had introduced a notion of completely regular spaces using the L-unit interval $(I_L, \Im)$ as is denoted in [10].

Definition 5.1 [16] An L-topological space (X, τ) is called *completely regular in sense of Hutton* if for any $f \in \tau$, there exists a collection $(g_\alpha)_{\alpha \in L}$ in L^X such that $f = \bigvee\limits_{\alpha \in L} g_\alpha$ and there exists an L-continuous mapping $g : (X, \tau) \to (I_L, \Im)$ such that

$$g_\alpha(y) \leq g(y)(1-) = \bigwedge_{t<1} g(y)(t) \leq g(y)(0+) = \bigvee_{s>0} g(y)(s) \leq f(y)$$

for all $y \in X$.

The following proposition show that our completely regular spaces are more general than the completely regular spaces in sense of Hutton.

Proposition 5.1 *Every completely regular space in sense of Hutton is completely regular in our sense.*

Proof. Let $x \notin F$, $F \in \tau'$ hold. Then $\chi_{F'} \in \tau$ and $\chi_{F'}(x) = 1$ and hence $\chi_{F'}(x) \geq \alpha$ for all $\alpha \in L$ and $\chi_{F'} = \bigvee\limits_{x \in F', \alpha \in L} x_\alpha$. That is, there exists for all $x \in F'$ a family $(x_\alpha)_{\alpha \in L}$ such that $\chi_{F'} = \bigvee\limits_{\alpha \in L} x_\alpha$. Since (X, τ) is a completely regular space in sense of Hutton, then there exists an L-continuous mapping $g : (X, \tau) \to (I_L, \Im)$ such that $x_\alpha(y) \leq g(y)(1-) \leq g(y)(0+) \leq \chi_{F'}(y)$ for all $y \in X$. If $y \in F$, then we get $0 \leq g(y)(1-) \leq g(y)(0+) \leq 0$, which means $g(y) = \overline{0}$ for all $y \in F$. In case of $y = x$, we get $x_\alpha(x) = \alpha \leq g(x)(1-) \leq g(x)(0+) \leq 1$ for all $\alpha \in L$, and this means $g(x)(s) = 1$ for all $s < 1$, and hence $g(x) = \overline{1}$. Thus, (X, τ) is a completely regular space in our sense. □

We have the following counter example.

Example 5.1 Let $X = \{x, y\}$ with $x \neq y$ and let $\tau = \{\overline{0}, \overline{1}, x_1, x_{\frac{1}{2}}, x_1 \vee y_{\frac{1}{2}}, x_{\frac{1}{2}} \vee y_1, x_{\frac{1}{2}} \vee y_{\frac{1}{2}}\}$. Then, $\tau' = \{\overline{0}, \overline{1}, y_1, x_{\frac{1}{2}}, y_{\frac{1}{2}}, x_{\frac{1}{2}} \vee y_{\frac{1}{2}}, x_{\frac{1}{2}} \vee y_1\}$ and there is only the case $x \notin \{y\} \in \tau'$ to be studied.

Since the mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \neq x$ is L-continuous, then (X, τ) is a completely regular space in our sense, and moreover it is also GT_1. Hence, (X, τ) is a $GT_{3\frac{1}{2}}$-space.

Since $x_{\frac{1}{2}} \in \tau$ and $x_{\frac{1}{2}} = \bigvee_{\alpha \in L} (\overline{\frac{1}{2}} \wedge x_\alpha)$, then there exists a collection $(g_\alpha)_{\alpha \in L} = (\overline{\frac{1}{2}} \wedge x_\alpha)_{\alpha \in L}$ such that $x_{\frac{1}{2}} = \bigvee_{\alpha \in L} g_\alpha$.

Now, for any L-continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \neq x$ we get that the inequality

$$g_\alpha(z) \leq f(z)(1-) \leq f(z)(0+) \leq x_{\frac{1}{2}}(z)$$

holds only when $z = y$ but at $z = x$ we get $(\frac{1}{2} \wedge \alpha) \leq 1 \leq \frac{1}{2}$ which is a contradiction and hence (X, τ) is not a completely regular space in sense of Hutton.

The relation between our completely regular spaces and the completely regular spaces in sense of Katsaras. In the following we study the relation between our completely regular spaces and the completely regular spaces in sense of Katsaras ([20]).

Definition 5.2 [20] An L-topological space (X, τ) is called *completely regular in sense of Katsaras* if for each open L-set μ in X and every $x \in X$ with $\mu(x) > \alpha$, $\alpha \in L_0$, there exists an L-continuous mapping $g : (X, \tau) \to (I_L, \Im)$ such that $g(y)(0+) \leq \mu(y)$ and $g(y)(1-) > \alpha$ for all $y \in X$.

Proposition 5.2 *Every completely regular space in sense of Katsaras is completely regular in our sense.*

Proof. Let $x \notin F$, $F \in \tau'$. Then $F' \in \tau$ and this means $\chi_{F'}(x) = 1 \geq \alpha$ for all $\alpha \in L$. From that (X, τ) is a completely regular space in sense of Katsaras, it follows that there exists an L-continuous mapping $g : (X, \tau) \to (I_L, \Im)$ such that $\bigvee_{t>0} g(y)(t) \leq \chi_{F'}(y)$ and $\bigwedge_{s<1} g(x)(s) > \alpha$ for all $y \in X$. If $y \in F$, then we get $\bigvee_{t>0} g(y)(t) \leq 0$, that is, $g(y)(t) = 0$ for all $t > 0$ and then $g(y) = \overline{0}$. At $y = x$, we get

$\bigwedge\limits_{s<1} g(x)(s) > \alpha$ for all $\alpha \in L$, and then $g(x) = \overline{1}$. Thus (X, τ) is completely regular in our sense. □

The following counter example introduce a completely regular space in our sense which is not completely regular in sense of Katsaras.

Example 5.2 Taking the same example as in Example 5.1, we get (X, τ) is a $GT_{3\frac{1}{2}}$-space.

Also, for any L-continuous mapping $f : (X, \tau) \to (I_L, \Im)$ with $f(x) = \overline{1}$ and $f(y) = \overline{0}$, we shall consider $x_{\frac{1}{2}} \in \tau$ with $x_{\frac{1}{2}}(x) = \frac{1}{2} > 0$ (that is, there is some $\alpha = \frac{1}{2} \in L$ such that $x_{\frac{1}{2}}(x) = \alpha > 0$). Then $f(z)(1-) = \bigwedge\limits_{t<1} f(z)(t) > \frac{1}{2}$ only if $z = x$ and it is not true if $z = y$. Also, $f(z)(0+) = \bigvee\limits_{s>0} f(z)(s) \le x_{\frac{1}{2}}(z)$ is true for $z = y$ but it is not true for $z = x$. Thus (X, τ) is not completely regular in sense of Katsaras.

The relation between the $GT_{3\frac{1}{2}}$-spaces and the $FT_{3\frac{1}{2}}$-spaces. In the following we introduce the relation between the $FT_{3\frac{1}{2}}$-spaces defined by Kandil and El-Shafee in [17] and our notion of $GT_{3\frac{1}{2}}$-spaces.

Definition 5.3 [17] An L-topological space (X, τ) is called an FT_1-*space* if for all $x, y \in X$ with $x \neq y$ we have $x_\alpha \, \overline{q} \, \mathrm{cl}_\tau y_\beta$ and $\mathrm{cl}_\tau x_\alpha \, \overline{q} \, y_\beta$ for all $\alpha, \beta \in L$.

Definition 5.4 [17] An L-topological space (X, τ) is called *completely regular in sense of Kandil and El-Shafee* if $x_\alpha \, \overline{q} \, f$, $f \in \tau'$ implies that there exists an L-continuous mapping $g : (X, \tau) \to (I_L, \Im)$ such that $g(y)(0+) \le f'(y)$ and $g(y)(1-) \ge x_\alpha(y)$ for all $y \in X$ and $\alpha \in L$. (X, τ) is called an $FT_{3\frac{1}{2}}$-*space* if it is an FT_1-space and a completely regular space in sense of Kandil and El-Shafee.

We have shown in [2] that our GT_1-spaces are more general than the FT_1-spaces.

Proposition 5.3 [2] *Every FT_1-space is a GT_1-space.*

Using Proposition 5.3, we prove in the following proposition that our $GT_{3\frac{1}{2}}$-spaces are more general than the $FT_{3\frac{1}{2}}$-spaces .

Proposition 5.4 *Every $FT_{3\frac{1}{2}}$-space is a $GT_{3\frac{1}{2}}$-space.*

Proof. Let (X, τ) be an $FT_{3\frac{1}{2}}$-space. Then it is GT_1 from Proposition 5.3. Now, let $x \notin F$ and $F \in \tau'$. Then $x \in F'$ and $x_1 \,\overline{q}\, \chi_F$. Thus there exists an L-continuous mapping $g : (X, \tau) \to (I_L, \Im)$ such that $g(y)(0+) \leq \chi_{F'}(y)$ and $g(y)(1-) \geq x_1(y)$ for all $y \in X$. At $y \in F$, we get $0 \leq g(y)(1-) \leq g(y)(0+) \leq 0$, that is, $g(y)(s) = 0$ for all $s > 0$, and then $g(y) = \overline{0}$ for all $y \in F$. In case of $y = x$, we get $1 \leq g(x)(1-) \leq g(x)(0+) \leq 1$, and then $g(x)(s) = 1$ for all $s < 1$, that is, $g(x) = \overline{1}$. Hence, the space (X, τ) is completely regular in our sense and thus it is a $GT_{3\frac{1}{2}}$-space. □

Now, we have the following counter example.

Example 5.3 Let (X, τ) be as in Example 5.1, that is, $X = \{x, y\}$ and $\tau = \{\overline{0}, \overline{1}, x_1, x_{\frac{1}{2}}, x_1 \vee y_{\frac{1}{2}}, x_{\frac{1}{2}} \vee y_1, x_{\frac{1}{2}} \vee y_{\frac{1}{2}}\}$. Then it is a $GT_{3\frac{1}{2}}$-space.

Now, for $g = x_{\frac{1}{2}} \in \tau'$, we get $x_{\frac{1}{2}} \,\overline{q}\, g$ and $g' = x_{\frac{1}{2}} \vee y_1 \in L^X$. Hence, for any L-continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \neq x$ we get $x_{\frac{1}{2}}(z) \leq f(z)(1-) = \bigwedge_{t<1} f(z)(t)$ holds for all $z \in X$. But, $g'(z) = (x_{\frac{1}{2}} \vee y_1)(z) \geq f(z)(0+) = \bigvee_{s>0} f(z)(s)$ is true for $z = y$ and it is not true for $z = x$. Thus the space (X, τ) is not $FT_{3\frac{1}{2}}$-space.

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